FIAN/TD/96-02
hep-th/9602048
January 1996

Pseudoclassical description of higher spins in $2+1$ dimensions

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Abstract

Pseudoclassical supersymmetric model to describe massive particles with higher spins (integer and half-integer) in $2+1$ dimensions is proposed. The quantization of the model leads to the minimal (with only one polarization state) quantum theory. In particular, the Bargmann-Wigner type equations for higher spins arise in course of the canonical quantization. The cases of spin one-half and one are considered in detail. Here one gets Dirac particles and Chern-Simons particles respectively. A relation with the field theory is discussed. On the basis of the model proposed, and using dimensional reduction considerations, a model to describe Weyl particles with higher spins in $3+1$ dimensions is constructed.

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1 Introduction

In this paper we present a pseudoclassical supersymmetric model to describe massive particles with higher spins (integer and half-integer) in 2+1 dimensions. Besides a pure theoretical interest to complete the theory of relativistic particles, there is a direct relation to the $2+1$ field theory [1], which attracts in recent years great attention due to various reasons: e.g. nontrivial topological properties, and especially the possibility of the existence of particles with fractional spins and exotic statistics (anyons), having probably applications to fractional Hall effect, high-T_c superconductivity and so on [2]. The well-known pseudoclassical supersymmetric model for Dirac (spin one-half) particles in $3+1$ dimensions was studied in numerous papers [3]. Generalizations of the model for particles with arbitrary spins in such dimensions, for Weyl particles, and so on, one be found, for example, in [4, 5, 6]. Attempts to extend the pseudoclassical description to arbitrary odd-dimensional case had met some problems, which are connected with the absence of an analog of γ^5-matrix in odd-dimensions. For instance, the direct dimensional reduction of the $3+1$ spinning particle action (standard action) to $2+1$ dimensions does not reproduce a minimal quantum theory of the spinning particle in $2+1$ dimensions, which has to provide only one value of the spin projection ($1/2$ or $-1/2$). In papers [7] two modifications of the standard action were proposed to get such a minimal theory. However, the first action [7] is in fact classically equivalent to the standard action in $2+1$ dimensions and does not provide required quantum properties in course of the canonical and path integral quantization. Moreover, it is P- and T- invariant, so that an anomaly is present. Another one does not obey gauge supersymmetries and therefore loses the main attractive features in such kind of models, which allow one to treat them as prototypes of superstrings or some modes in the superstring theory. In [8] we succeeded to write a new action to describe spin one-half in $2+1$ dimensions, which reproduces the minimal quantum theory of this spin after quantization. Here we propose a model to describe all higher spins (integer and half-integer) in $2+1$ dimensions. The action of the model is invariant under three kinds of gauge transformations: reparametrizations and two supertransformations. It is P- and T-noninvariant in full agreement with the expected properties of the minimal theory of higher spins in $2+1$ dimensions. First, we quantize the general model canonically, using a simple realization in a Fock space, to demonstrate that the minimal quantum theory of higher spins is reproduced. Then we consider the cases of spin one-half and spin one in detail. In the first case we present both canonical and Dirac quantizations to get Dirac equation in $2+1$ dimensions. It turns out that in the case of spin one the model proposed describes Chern-Simons particles. In particular, one can see that the equations of the topologically massive gauge theory are reproduced in course of the quantization. Then, in the general case, we present a realization of the canonical quantization , which corresponds to the Bargmann-Wigner type formulation of higher spins theory in. A relation of the quantum mechanics constructed with the field theory is discussed. In the end of the paper we demonstrate that on the basis of the model proposed, and using dimensional reduction considerations, a model to describe Weyl particles with higher spins in $3+1$ dimensions can be constructed.

2 The action of the model, symmetries, and Hamiltonian formulation

Consider pseudoclassical action of the form

$$S=\int_0^1\left\{-\frac{z^2}{2e}-\frac{m^2}{2}e-\sum_{a=1}^{N}\left[sm\left(\frac{\kappa_a}{2}+i\psi_a^3\chi_a\right)+i\psi_{an}\dot{\psi}_a^n\right]\right\}d\tau=\int_0^1 Ld\tau\ ,$$
$$z^\mu=\dot{x}^\mu+i\sum_{a=1}^{N}\left(\varepsilon^{\mu\nu\lambda}\psi_{a\nu}\psi_{a\lambda}\kappa_a-\psi_a^\mu\chi_a\right);\ N=1,2,\ldots;\quad s=\pm1\ , \tag{1}$$

where the Greek (Lorentz) indices μ, ν, λ, run over 0, 1, 2, whereas the Latin ones n,m, run over $0,1,2,3$, one supposes summation over the repeated Greek and Latin (n,m) indices (but not over the index a); $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1)$, $\eta_{mn}=\mathrm{diag}(1,-1,-1,-1)$; $x^\mu,\ e,\ \kappa_a$ are even and $\psi_{an},\ \chi_a$ are odd variables; $\varepsilon^{\mu\nu\lambda}$ is the totally antisymmetric tensor density of Levi-Civita in $2+1$ dimensions. We suppose that x^μ and $\psi_{a\mu}$ are $2+1$ Lorentz vectors and e, κ_a, ψ_a^3, χ_a are scalars, so that the action (1) is invariant under the restricted Lorentz transformations (but is not P- and T-invariant). It is invariant under the reparametrizations and under other two types of gauge transformations, one of which is a supergauge transformation:

$$\delta x^\mu=\dot{x}^\mu\xi\ ,\quad \delta e=\frac{d}{d\tau}(e\xi)\ ,\quad \delta\psi_{an}=\dot{\psi}_{an}\xi\ ,\quad \delta\chi_a=\frac{d}{d\tau}(\chi_a\xi)\ ,\quad \delta\kappa_a=\frac{d}{d\tau}(\kappa_a\xi)\ ; \tag{2}$$
$$\delta x^\mu=i\sum_{a=1}^{N}\psi_a^\mu\epsilon_a\ ,\quad \delta e=i\sum_{a=1}^{N}\chi_a\epsilon_a\ ,\quad \delta\psi_a^\mu=\frac{z^\mu}{2e}\epsilon_a\ ,\quad \delta\psi_a^3=s\frac{m}{2}\epsilon_a,\quad \delta\chi_a=\dot{\epsilon}_a\ ,\quad \delta\kappa_a=0 \tag{3}$$
$$\delta x^\mu=-i\sum_{a=1}^{N}\varepsilon^{\mu\nu\lambda}\psi_{a\nu}\psi_{a\lambda}\theta_a\ ,\quad \delta\psi_a^\mu=\frac{1}{e}\varepsilon^{\mu\nu\lambda}z_\nu\psi_{a\lambda}\theta_a\ ,\quad \delta\kappa_a=\dot{\theta}_a,\quad \delta e=\delta\psi_a^3=\delta\chi_a=0\ , \tag{4}$$

where $\xi(\tau),\ \theta_a(\tau)$ are even, and $\epsilon_a(\tau)$ are odd parameters.

Going over to the Hamiltonian formulation, we introduce the canonical momenta,

$$\pi_\mu=\frac{\partial L}{\partial\dot{x}^\mu}=-\frac{1}{e}z_\mu\ ,\quad P_e=\frac{\partial L}{\partial\dot{e}}=0\ ,\quad P_{\chi_a}=\frac{\partial_r L}{\partial\dot{\chi}_a}=0\ ,$$
$$P_{\kappa_a}=\frac{\partial L}{\partial\dot{\kappa}_a}=0\ ,\quad P_{an}=\frac{\partial_r L}{\partial\dot{\psi}_a^n}=-i\psi_{an}\ . \tag{5}$$

It follows from (5) that there exist primary constraints $\Phi^{(1)}=0$,

$$\Phi_1^{(1)}=P_e,\ \Phi_2^{(1)}=P_{\chi_a},\ \Phi_3^{(1)}=P_{\kappa_a},\ \Phi_4^{(1)}=P_{an}+i\psi_{an}\ . \tag{6}$$

Constructing the total Hamiltonian $H^{(1)}$, according to the standard procedure [9, 10], we get $H^{(1)}=H+\lambda_A\Phi_A^{(1)}$, where

$$H=-\frac{e}{2}(\pi^2-m^2)+i\sum_{a=1}^{N}(\pi_\mu\psi_a^\mu+sm\psi_a^3)\chi_a-i\sum_{a=1}^{N}(\varepsilon^{\mu\nu\lambda}\pi_\mu\psi_{a\nu}\psi_{a\lambda}+\frac{i}{2}sm)\kappa_a\ . \tag{7}$$

From the consistency conditions $\dot{\Phi}^{(1)}=\{\Phi^{(1)},H^{(1)}\}=0$ one can find secondary constraints $\Phi^{(2)}=0$,

$$\Phi_1^{(2)}=\pi^2-m^2,\ \Phi_2^{(2)}=\pi_\mu\psi_a^\mu+sm\psi_a^3\ ,\quad \Phi_3^{(2)}=\varepsilon^{\mu\nu\lambda}\pi_\mu\psi_{a\nu}\psi_{a\lambda}+\frac{i}{2}sm\ , \tag{8}$$

and determine λ, which correspond to the primary constraints $\Phi_4^{(1)}$. No more secondary constraints arise from the consistency conditions and the Lagrangian multipliers, which correspond to the primary constraints $\Phi_i^{(1)},\ i=1,2,3$, remain undetermined. The Hamiltonian (7) is proportional to the constraints. One can go over from the initial set of constraints $\Phi^{(1)},\Phi^{(2)}$ to the equivalent one $\Phi^{(1)},\tilde{\Phi}^{(2)}$, where $\tilde{\Phi}^{(2)}=\Phi^{(2)}\left(\psi\rightarrow\tilde{\psi}=\psi+\frac{i}{2}\Phi_4^{(1)}\right)$. The new set of constraints can be explicitly divided in a set of the first-class constraints, which are $(\Phi_i^{(1)},\ i=1,2,3,\ \tilde{\Phi}^{(2)})$ and in a set of second-class constraints $\Phi_4^{(1)}$.

Calculating the total angular momentum tensor $M_{\mu\nu}$, which corresponds to the action (1), we get

$$M_{\mu\nu}=L_{\mu\nu}+S_{\mu\nu},\ L_{\mu\nu}=x_\mu\pi_\nu-x_\nu\pi_\mu,\ S_{\mu\nu}=i\sum_{a=1}^{N}[\psi_{a\mu},\psi_{a\nu}]\ . \tag{9}$$

The dual vector $J^\mu=\frac{1}{2}\varepsilon^{\mu\nu\lambda}M_{\nu\lambda}$ together with the momentum π_μ are generators of the $2+1$ Poincare algebra. The Pauli-Lubanski scalar W,

$$W=\pi_\mu J^\mu=\pi_\mu S^\mu,\ \ S^\mu=\frac{1}{2}\varepsilon^{\mu\nu\lambda}S_{\nu\lambda}, \tag{10}$$

specifies the helicity (spin) of the particles and similar to π^2 is a Casimir operator in the case of consideration.

To quantize the theory canonically one has to impose as much as possible supplementary gauge conditions to the first-class constraints. In the case under consideration, it turns out to be possible to impose the gauge conditions to all the first-class constraints, excluding the constraints $\tilde{\Phi}_3^{(2)}$. These constraints are quadratic in the grassmannian variables. On the one hand, that circumstance makes it difficult to impose a conjugated gauge condition, on the other hand, imposing these constraints on state vectors does not create problems with the Hilbert space definition since the corresponding operators of constraints have a discrete spectrum. Thus, we shall treat only the constraints $\tilde{\Phi}_3^{(2)}$ in sense of the Dirac method, fixing only the gauge freedom, which corresponds to two types of gauge transformations (2) and (3). As a result we remain only with the first-class constraints, which are the reduction of $\Phi_3^{(2)}$ to the rest of constraints and gauge conditions. They can be used to specify the physical states. All the second-class constraints form the Dirac brackets. The following gauge conditions $\Phi^G=0$ can be imposed: $\Phi_1^G=e+\zeta\pi_0^{-1}$, $\Phi_2^G=\chi_a$, $\Phi_3^G=\kappa_a$, $\Phi_4^G=x_0-\zeta\tau$, $\Phi_5^G=\psi_a^0$, where $\zeta=-\mathrm{sign}\,\pi^0$ (The gauge $x_0-\zeta\tau=0$ was first proposed in [11, 10] as a conjugated gauge condition to the constraint $\pi^2-m^2=0$, see there a detailed discussion of this gauge). Using the consistency conditions $\dot{\Phi}^G=0$, one can determine the Lagrangian multipliers, which correspond to the primary constraints $\Phi_i^{(1)},\ i=1,2,3$. To go over to a time-independent set of constraints (to use the standard scheme of quantization [9] without modifications [10], which are necessary if the constraints depend on time explicitly), we introduce the variable x_0', $x_0'=x_0-\zeta\tau$, instead of x_0, without changing the rest of the variables. That is a canonical transformation in the space of all the variables with the generating function $W=x_0\pi_0'+\tau|\pi_0'|+W_0$, where W_0 is the generating function of the identity transformation with respect to all the variables except x^0 and π_0. The transformed Hamiltonian $H^{(1)\prime}$ is of the form

$$H^{(1)\prime}=\omega+\{\Phi\},\ \ \omega=\sqrt{\vec{\pi}^2+m^2},\ \ \vec{\pi}^2=\pi_k\pi_k,\ k=1,2\ , \tag{11}$$

where $\{\Phi\}$ are terms proportional to the constraints and ω is the physical Hamiltonian. Now all the constraints of the theory can be presented in the following equivalent form: $K = 0$, $\phi = 0$, $T = 0$, where

$$
\begin{aligned}
&K = (e - \omega^{-1}\ ,\ P_e\ ;\ \ \chi_a\ ,\ P_{\chi_a}\ ;\ \ \kappa_a\ ,\ P_{\kappa_a}\ ;\ \ x_0'\ ,\ |\pi_0| - \omega\ ;\ \ \psi_a^0\ ,\ P_{a0}),\ a = 1, \ldots, N\ ;\\
&\phi = (\pi_k \psi_a^k + sm\psi_a^3,\ P_{ad} + i\psi_{ad})\ ,\ \ k = 1, 2\ ,\ \ d = 1, 2, 3\ ;\\
&T_a = \zeta\omega[\psi_a^2, \psi_a^1] + \frac{i}{2} sm\ .
\end{aligned}
\tag{12}
$$

Here K and ϕ are second-class constraints, whereas T_a are first-class ones. Besides, the set K has the so called special form [10]. In this case, if we eliminate the variables e, P_e, χ_a, P_{χ_a}, κ_a, P_{κ_a}, x_0', $|\pi_0|$, ψ_a^0, and P_{a0}, using the constraints $K = 0$, the Dirac brackets with respect to all the second-class constraints (K, ϕ) reduce to ones with respect to the constraints ϕ only. Thus, in fact, we can only consider the variables x^k, π_k, ζ, ψ_a^k, P_{ak}, $k = 1, 2$, and two sets of constraints: the second-class ones ϕ and the first-class ones T. Nonzero Dirac brackets for the independent variables are

$$
\begin{aligned}
&\{x^k, \pi_r\}_{D(\phi)} = \delta_{kr}\ ,\ \ \{x^k, x^r\}_{D(\phi)} = \frac{i}{\omega^2}\sum_{a=1}^{N}[\psi_a^k, \psi_a^r]\ ,\ \ \{x^k, \psi_a^r\}_{D(\phi)} = -\frac{1}{\omega^2}\psi_a^k \pi_r\ ,\\
&\{\psi_a^k, \psi_b^r\}_{D(\phi)} = -\frac{i}{2}(\delta_{kr} - \omega^{-2}\pi_k\pi_r)\delta_{ab}\ ,\ \ k, r = 1, 2\ .
\end{aligned}
\tag{13}
$$

The Dirac brackets between J^μ, π_μ, W, and π^2 have the form

$$
\begin{aligned}
&\{J^\mu, J^\nu\}_{D(\phi)} = \varepsilon^{\mu\nu\lambda} J_\lambda,\ \{\pi_\mu, J_\nu\}_{D(\phi)} = \varepsilon_{\mu\nu\lambda}\pi^\lambda,\\
&\{\pi_\mu, W\}_{D(\phi)} = \{J^\mu, W\}_{D(\phi)} = \{J^\mu, \pi^2\}_{D(\phi)} = 0\ .
\end{aligned}
\tag{14}
$$

That means the $2+1$ Poincare algebra with the Casimir operators $\hat{\pi}^2$ and $\hat{W}$,

$$
[\hat{J}^\mu, \hat{J}^\nu] = i\varepsilon^{\mu\nu\lambda}\hat{J}_\lambda,\ [\hat{\pi}_\mu, \hat{J}_\nu] = i\varepsilon_{\mu\nu\lambda}\hat{\pi}^\lambda\ ,
\tag{15}
$$

is reproduced on the quantum level.

3 Quantization

3.1 Preliminary consideration

To verify right away that the model proposed reproduces particles with higher spins in $2+1$ dimensions after quantization, we consider first a simple realization in a Fock space. Then in the next subsections we present different realization, which, however, has more close relation to the field theory.

Let us go over to new variables whose Dirac brackets have a simple form. Namely, introduce new even variables X^k and odd variables θ_a^k according to the formulas

$$
X^k = x^k + \frac{i\pi_r}{m(\omega + m)}\sum_{a=1}^{N}[\psi_a^k, \psi_a^r]\ ,\ \ \theta_a^k = \psi_a^k + \frac{\pi_k(\omega - m)}{m\vec{\pi}^2}\pi_r\psi_a^r\ .
\tag{16}
$$

Using the brackets (13), we get for the new variables

$$\{X^k,\pi_r\}_{D(\phi)}=\delta_{kr}\ ,\ \ \{X^k,X^r\}_{D(\phi)}=\{X^k,\theta_a^r\}_{D(\phi)}=\{\pi_k,\theta_a^r\}_{D(\phi)}=0\ ,$$
$$\{\theta_a^k,\theta_b^r\}_{D(\phi)}=-\frac{i}{2}\delta_{kr}\delta_{ab}\ ,\quad k,r=1,2\ . \tag{17}$$

The variables $X^k,\ \pi_k,\ \zeta,\ \theta_a^k$ are independent with respect to the second-class constraints. Thus, we remain only with the first-class constraints (12), which being written in terms of the new variables have the form

$$T_a=m\left(\zeta[\theta_a^2,\theta_a^1]+\frac{i}{2}s\right)=0\ . \tag{18}$$

The Dirac brackets (17) define the commutation relations between the correspondent operators. The nonzero commutators (anticommutators) are

$$\left[\hat{X}^k,\hat{\pi}_r\right]=i\delta_{kr}\ ,\ \ \left[\hat{\theta}_a^k,\hat{\theta}_b^r\right]_+=\frac{1}{2}\delta_{kr}\delta_{ab}\ . \tag{19}$$

We assume as usual [11, 10] the operator $\hat{\zeta}$ to have the eigenvalues $\zeta=\pm1$ by analogy with the classical theory, so that $\hat{\zeta}^2=1$, and also we assume the equations of the second-class constraints $\hat{\phi}=0$. Then one can realize the algebra of all the independent operators in a Hilbert space $\mathcal{R}$, whose elements $\mathbf{\Psi}\in\mathcal{R}$ are two-component columns dependent on $\mathbf{x}=(x^k),\ k=1,2,$

$$\mathbf{\Psi}=\begin{pmatrix}\mathbf{f}_+(\mathbf{x})\\ \mathbf{f}_-(\mathbf{x})\end{pmatrix}\ ,\ \ \hat{\zeta}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ ,\ \ \hat{X}^k=x^k\ ,\ \ \hat{\pi}_d=-i\partial_d\ , \tag{20}$$

and $\mathbf{f}_+(\mathbf{x})$ and $\mathbf{f}_-(\mathbf{x})$ are $\mathbf{x}$ dependent vectors of a Fock space, which is constructed on the base of the fermionic operators (c_a^+,c_a) of creation and annihilation,

$$\hat{c}_a=\hat{\theta}_a^1+i\hat{\theta}_a^2\ ,\ \ \hat{c}_a^+=\hat{\theta}_a^1-i\hat{\theta}_a^2\ ,$$
$$[\hat{c}_a,\hat{c}_b^+]_+=\delta_{ab},\ [\hat{c}_a,\hat{c}_b]_+=[\hat{c}_a^+,\hat{c}_b^+]_+=0. \tag{21}$$

The operators $\hat{T}_a$ correspondent to the first-class constraint (18) have the form

$$\hat{T}_a=im\hat{\zeta}\left(\hat{n}_a-\lambda\right),\ \ \hat{n}=\hat{c}_a^\dagger\hat{c}_a\ ,\ \ \lambda=\frac{1-\hat{\zeta}s}{2}\ . \tag{22}$$

These operators specify physical states:

$$\hat{T}_a\mathbf{\Psi}=0\Leftrightarrow\hat{n}_a\mathbf{f}_\zeta=\delta_{-s,\zeta}\mathbf{f}_\zeta,\ \ \zeta=\pm\ . \tag{23}$$

Thus, $\mathbf{f}_\zeta$ are proportional to the vacuum vector $|0>$ in the Fock space, $\hat{c}_a|0>=0$, or to the vector $|N>=\hat{c}_1^+\dots\hat{c}_N^+|0>$. On the other hand, the state vectors $\mathbf{\Psi}$ have to obey the Schrödinger equation, which defines their “time” dependence, $(i\partial/\partial\tau-\hat{\omega})\mathbf{\Psi}=0$, where the quantum Hamiltonian $\hat{\omega}$ corresponds to the classical one ω, (11). Introducing the physical time $x^0=\zeta\tau$ instead of the parameter τ [11, 10], we can rewrite the Schrödinger equation in the following form

$$(i\partial/\partial x^0-\hat{\zeta}\hat{\omega})\mathbf{\Psi}(x)=0,\ \ \hat{\omega}=\sqrt{\hat{\vec{\pi}}^2+m^2},\ \ x=(x^0,\ \mathbf{x}). \tag{24}$$

Together with the eq. (23) that leads to the following structure of the physical state vectors

$$\mathbf{\Psi}=\left(\begin{array}{c} f_+(x)|0> \\ f_-(x)|N> \end{array}\right),\quad i\frac{\partial}{\partial x^0}f_\pm(x)=\pm\hat\omega f_\pm(x). \tag{25}$$

We interpret $f_+(x)|0>$ as the wave function of a particle and $f_-^*(x)|N>$ as the wave function of an antiparticle. Both particles and antiparticles have only one polarization state in full agreement with the group analysis [12].

What is the helicity (spin) of the particles and antiparticles obtained? To answer this question let us use the Pauli–Lubanski operator $\hat W$ which corresponds to the scalar (10). In the gauge selected and in the realization in question, it has the form

$$\hat W=i\hat\zeta m\sum_{a=1}^N[\hat\theta_a^2,\hat\theta_a^1]=\hat\zeta m\left(\frac{N}{2}-\hat n\right),\quad \hat n=\sum_{a=1}^N\hat n_a. \tag{26}$$

One can easily see that the state vectors (25) are eigenfunctions for the operator (26),

$$\hat W\mathbf{\Psi}=m\frac{sN}{2}\mathbf{\Psi}\ . \tag{27}$$

The latter means that the spin of the particles and antiparticles is equal to. Thus, we see that the action (1) describes particles with helicity (spin) $sN/2$. It is important to compare the quantum mechanics constructed with the field theory. To this end, however, another realization is more convenient. We consider it in the two next subsections.

3.2 Spin one-half case

Let us consider particles with spin one-half. The corresponding model follows from the general expression (1) at $N=1$. The canonical quantization considered above gives a quantum mechanics, which completely corresponds to our ideas about the Dirac particles in such dimensions. To get a relation with the corresponding field theory let us consider a special realization for initial variables x^k and ψ^k.

It follows from the Dirac brackets (13) for $N=1$ that nonzero commutation relations have the form

$$\begin{aligned}&[\hat x^k,\hat\pi_r]=i\delta_{kr}\ ,\quad [\hat x^k,\hat x^r]=-\frac{1}{\hat\omega^2}[\hat\psi^k,\hat\psi^r]\ ,\quad [\hat x^k,\hat\psi^r]=-\frac{i}{\hat\omega^2}\hat\psi^k\hat\pi_r\ ,\\ &[\hat\psi^k,\hat\psi^r]_+=\frac12(\delta_{kr}-\hat\omega^{-2}\hat\pi_k\hat\pi_r)\ ,\quad k,r=1,2\ .\end{aligned}$$

One can realize the algebra of all the independent operators in a Hilbert space $\mathcal{R}$ whose elements $\mathbf{\Psi}\in\mathcal{R}$ depend on $\mathbf{x}=(x^d)$, $d=1,2$, and have a form

$$\mathbf{\Psi}(\mathbf{x})=\left(\begin{array}{c}\Psi^{(+)}(\mathbf{x})\\ \Psi^{(-)}(\mathbf{x})\end{array}\right)\ ,$$

where $\Psi^{(\pm)}(\mathbf{x})$ are two-component spinors, $\Psi^{(\pm)}_\alpha(\mathbf{x}),\ \alpha=1,2$. In this space

$$\hat x^k=x^k+\Delta\hat x^k,\quad \Delta\hat x^k=\frac{1}{2\hat\omega^2}\varepsilon^{kr}\left(\hat\pi_r\Sigma^3-sm\Sigma^r\right)\ ,$$

$$\hat{\pi}_k = -i\partial_k \ , \ \ \hat{\zeta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \ \Sigma^k = \mathrm{diag}(\sigma^k, \ \sigma^k) \ ,$$

$$\hat{\psi}^k = \frac{1}{2}\left[\Sigma^k - \frac{\hat{\pi}_k}{\hat{\omega}^2}(\hat{\pi}_r\Sigma^r + sm\Sigma^3)\right] , \ \ k,r = 1,2 \ , \tag{28}$$

where σ^k are Pauli matricies. Constructing the operator $\hat{T}$ according to the first-class constraint (12) at $N = 1$, we specify the physical states,

$$\hat{T}\mathbf{\Psi} = 0 \ , \ \ \hat{T} = \frac{ism\Sigma^3}{2\hat{\omega}}\hat{\zeta}\left[\hat{\zeta}\hat{\omega}\Sigma^3 + i\partial_1(i\Sigma^2) + i\partial_2(-i\Sigma^1) - sm\right] \ . \tag{29}$$

Besides, the Schrödinger equation (24) holds for these states. The combination of the latter equation with the condition (29) leads to the Dirac equation in $2+1$ dimensions,

$$(i\partial_\mu\gamma^\mu - sm)\Psi^{(\zeta)}(x) = 0 \ , \ \ \zeta = \pm \ , \tag{30}$$

where γ^μ are γ-matrices in $2+1$ dimensions,

$$\begin{aligned} &\gamma^0 = \sigma^3, \ \gamma^1 = i\sigma^2, \ \gamma^2 = -i\sigma^1, \\ &[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \ \ \gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\varepsilon^{\mu\nu\lambda}\gamma_\lambda \ , \\ &\gamma^{+\mu} = \gamma^0\gamma^\mu\gamma^0, \ C\gamma^\mu C = -\gamma^{T\mu}, \ C = \sigma^2. \end{aligned} \tag{31}$$

Calculating the operator $\hat{W}$, which corresponds to the Pauli-Lubanski scalar at $N = 1$, we get

$$\hat{W} = \frac{sm\Sigma^3\hat{\zeta}}{2\hat{\omega}}\left[sm - i\partial_1\gamma^1 - i\partial_2\gamma^2\right] . \tag{32}$$

Its action on the states, which obey the equations of motion (24), (29), gives the spin $s/2$ for the particles,

$$\hat{W}\mathbf{\Psi}(x) = m\frac{s}{2}\mathbf{\Psi}(x) \ . \tag{33}$$

One can also verify that the operators $\hat{M}_{\mu\nu}$, constructed according to the expression for the angular momentum tensor at $N = 1$, act on the mass shell as Lorentz transformations generators,

$$\hat{M}_{\mu\nu}\mathbf{\Psi}(x) = \left\{-i(x_\mu\partial_\nu - x_\nu\partial_\mu) - \frac{i}{4}\begin{pmatrix} [\gamma_\mu, \gamma_\nu] & 0 \\ 0 & [\gamma_\mu, \gamma_\nu] \end{pmatrix}\right\}\mathbf{\Psi}(x) \ . \tag{34}$$

It follows from the Schrödinger equation that $\Psi^{(\pm)}$ can be interpreted as positive and negative frequency solutions to this equation. Thus, a natural interpretation of the components $\Psi^{(\zeta)}(x)$ is the following: $\Psi^{(+)}(x)$ is the wave function of a particle with the spin $s/2$ and $\Psi^{*(-)}(x)$ is the wave function of an antiparticle with the spin $s/2$. Such an interpretation can be confirmed if we introduce in the model the interaction with an external electromagnetic field, namely, if we add to the Lagrangian of the model the following terms

$$-g\dot{x}^\mu A_\mu + igeF_{\mu\nu}\psi^\mu\psi^\nu \ ,$$

where g is the $U(1)$-charge. In this case the coupling constants with the external field in the equations for the wave functions have different sign, for particles g and for antiparticles $-g$.

It is also instructive to demonstrate that the canonical quantization considered is equivalent to the Dirac one, where the second–class constraints $\Phi_4^{(1)}$ define the Dirac brackets and therefore the commutation relations, whereas, all the first-class constraints, being applied to the state vectors, define physical states. Thus, here we will not impose explicitly any gauge conditions. For essential operators and nonzero commutation relations one can obtain in the case under consideration:

$$[\hat{x}^\mu, \hat{\pi}_\nu] = i\{x^\mu, \pi_\nu\}_{D(\Phi_4^{(1)})} = i\delta_\nu^\mu \ , \ \ [\hat{\psi}^n, \hat{\psi}^m]_+ = i\{\psi^n, \psi^m\}_{D(\Phi_4^{(1)})} = -\frac{1}{2}\eta^{nm} \ . \tag{35}$$

It is possible to construct a realization of the commutation relations (35) in a Hilbert space $\mathcal{R}$ whose elements $\mathbf{\Psi} \in \mathcal{R}$ are four–component columns dependent on x,

$$\mathbf{\Psi}(x) = \begin{pmatrix} \varphi(x) \\ \Psi(x) \end{pmatrix}, \ \ \hat{x}^\mu = x^\mu \ , \ \ \hat{\pi}_\mu = -i\partial_\mu \ , \ \ \hat{\psi}^n = \frac{i}{2}\Gamma^n, \tag{36}$$

where $\varphi(x)$ and $\Psi(x)$ are two-component columns, and Γ^n, $n = 0, 1, 2, 3$, are γ-matrices in $3+1$ dimensions, which we select in the spinor representation $\Gamma^0 = \mathrm{antidiag}(I, \ I)$, $\Gamma^i = \mathrm{antidiag}(\sigma^i, \ -\sigma^i)$, $i = 1, 2, 3$. According to the scheme of quantization chosen, the operators of the first-class constraints have to be applied to the state vectors to define the physical sector, namely, the physical states obey the equations $\hat{\Phi}^{(2)}\mathbf{\Psi}(x) = 0$, where $\hat{\Phi}^{(2)}$ are operators, which correspond to the constraints (8). Taken into account (36), one can write the equation $\hat{\Phi}_2^{(2)}\mathbf{\Psi}(x) = 0$ as

$$(i\partial_\mu\Gamma^\mu - sm\Gamma^3)\mathbf{\Psi}(x) = 0 \ \ \Longleftrightarrow \left\{ \begin{array}{l} (i\partial_\mu\gamma^\mu - sm)\Psi(x) = 0 \ , \\ (i\partial_\mu\gamma^{\dagger\mu} + sm)\varphi(x) = 0 \ . \end{array} \right. \tag{37}$$

Constructing the operator $\hat{\Phi}_1^{(2)}$ according to the classical function $\Phi_1^{(2)}$, we discover that the equation $\hat{\Phi}_1^{(2)}\mathbf{f} = 0$ is not independent, since in this case $\hat{\Phi}_1^{(2)} = (\hat{\Phi}_2^{(2)})^2$. The equation $\hat{\Phi}_3^{(2)}\mathbf{f}(x) = 0$ can be presented in the following form

$$\left(\frac{i}{2}\varepsilon^{\mu\nu\lambda}\partial_\mu\Gamma_\nu\Gamma_\lambda + ism\right)\mathbf{\Psi}(x) = 0 \ \ \Longleftrightarrow \ \left\{ \begin{array}{l} (i\partial_\mu\gamma^\mu - sm)\Psi(x) = 0 \ , \\ (i\partial_\mu\gamma^{\dagger\mu} - sm)\varphi(x) = 0 \ . \end{array} \right. \tag{38}$$

Combining eqs. (37) and (38), we get $\varphi(x) \equiv 0$ and $\Psi(x)$ obeys the $2+1$ Dirac equation

$$(i\partial_\mu\gamma^\mu - sm)\Psi(x) = 0 \ , \tag{39}$$

To interpret the quantum mechanics constructed one has to take into account the operator, which corresponds to the angular momentum tensor,

$$\hat{M}_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) - \frac{i}{4}\begin{pmatrix} [\gamma_\mu^\dagger, \gamma_\nu^\dagger] & 0 \\ 0 & [\gamma_\mu, \gamma_\nu] \end{pmatrix} \ .$$

Thus one can see that in fact the quantum mechanical states are described by the two component wave function $\Psi(x)$, which obeys the Dirac equation in $2+1$ dimensions and is transformed under the spinor representation of the corresponding Lorentz group.

3.3 Bargmann–Wigner type realization

By analogy with the canonical quantization presented above for spin one–half case one can fulfil a quantization for arbitrary higher spin, which leads to the Bargmann–Wigner type wave equations [16]. Let us depart from the Dirac brackets (13), which imply the following nonzero commutation relations

$$[\hat{x}^k, \hat{\pi}_r] = i\delta_{kr} \ , \quad [\hat{x}^k, \hat{x}^r] = -\frac{1}{\hat{\omega}^2}\sum_{a=1}^{N}[\hat{\psi}_a^k, \hat{\psi}_a^r] \ , \quad [\hat{x}^k, \hat{\psi}_a^r] = -\frac{i}{\hat{\omega}^2}\hat{\psi}_a^k\hat{\pi}_r \ ,$$

$$[\hat{\psi}_a^k, \hat{\psi}_b^r]_+ = \frac{1}{2}(\delta_{kr} - \hat{\omega}^{-2}\hat{\pi}_k\hat{\pi}_r)\delta_{ab} \ , \quad k, r = 1, 2 \ . \tag{40}$$

We can realize now the algebra of all the operators in a Hilbert space $\mathcal{R}$ whose elements $\mathbf{\Psi} \in \mathcal{R}$ depend on $\mathbf{x} = (x^d), d = 1, 2$, and have the form

$$\mathbf{\Psi}(\mathbf{x}) = \left(\begin{array}{c} \Psi^{(+)}(\mathbf{x}) \\ \Psi^{(-)}(\mathbf{x}) \end{array}\right) , \tag{41}$$

where each component $\Psi^{(\pm)}(\mathbf{x})$ has N spinor indices α, $\Psi^{(\pm)}(\mathbf{x}) = \Psi^{(\pm)}(\mathbf{x})_{\alpha_1\ldots\alpha_N}$, $\alpha_a = 1, 2$. In this space

$$\hat{x}^k = x^k + \sum_{a=1}^{N}\left(\prod_{j=1}^{a-1}\otimes 1\right)\otimes\Delta\hat{x}^k\otimes\left(\prod_{j=a+1}^{N}\otimes 1\right) ,$$

$$\hat{\pi}_k = -i\partial_k \ , \quad \hat{\zeta} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) ,$$

$$\hat{\psi}_a^k = \left(\prod_{j=1}^{a-1}\otimes 1\right)\otimes\hat{\psi}^k\otimes\left(\prod_{j=a+1}^{n}\otimes\frac{1}{\hat{\omega}}(\hat{\pi}_r\Sigma^r + sm\Sigma^3)\right) , \tag{42}$$

where the operators $\Delta\hat{x}^k$ and $\hat{\psi}^k$ are defined in (28). The operators $\hat{T}_a$, which correspond to the first–class constraints (12), have the form

$$\hat{T}_a = \left(\prod_{j=1}^{a-1}\otimes 1\right)\otimes\hat{T}\otimes\left(\prod_{j=a+1}^{N}\otimes 1\right) ,$$

where the operator $\hat{T}$ is defined in (29). They specify the physical space $\hat{T}_a\mathbf{\Psi} = 0$. Due to the Srödinger equation, which has still the form (24), the former equations imply that both components $\Psi^{(\pm)}$ obey the Dirac equation (30) for each spinor index,

$$(i\partial_\mu\gamma^\mu - sm)_{\alpha_a\alpha'_a}\Psi^{(\pm)}_{\alpha_1\ldots\alpha'_a\ldots\alpha_N}(x) = 0 \ , \quad \alpha_a = 1, 2, \quad a = 1, \ldots, N \ , \tag{43}$$

and therefore obey also the Klein–Gordon equation

$$(\Box + m^2)\Psi^{\pm}(x) = 0 \ . \tag{44}$$

The operator $\hat{\mathbf{W}}$, which correspond to the Pauli–Lubanski scalar (10), has the form

$$\hat{\mathbf{W}} = \sum_{a=1}^{N}\left(\prod_{j=1}^{a-1}\otimes 1\right)\otimes\hat{W}\otimes\left(\prod_{j=a+1}^{N}\otimes 1\right) ,$$

where $\hat{W}$ is defined by eq. (32). Its action on the mass shell is:

$$\hat{\mathbf{W}}\mathbf{\Psi}(x) = m\frac{sN}{2}\mathbf{\Psi}(x) \ .$$

Thus, the particles described by the states (41) have the helicity $sN/2$. The operators $\hat{\mathbf{M}}_{\mu\nu}$, correspondent to the angular momentum tensor reproduce the action of the Lorentz transformation generators on the mass shell,

$$\hat{\mathbf{M}}_{\mu\nu}\mathbf{\Psi}(x) = \left\{ -i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \sum_{a=1}^{N}\left(\prod_{j=1}^{a-1}\otimes 1\right)\otimes \hat{M}_{\mu\nu}\otimes\left(\prod_{j=a+1}^{N}\otimes 1\right)\right\}\mathbf{\Psi}(x) \ .$$

where $\hat{M}_{\mu\nu}$ is defined by eq. (34). Similar to the case of spin one–half, $\Psi^{(+)}(x)$ and $\Psi^{(-)}(x)$ are positive and negative frequency solutions to the wave equation, so that one can interpret $\Psi^{(+)}(x)$ as the wave function of a particle with spin $sN/2$ and $\Psi^{(-)*}(x)$ as that of an antiparticle with the same spin.

The equations (43) are (2+1) analog of the Bargmann–Wigner equations, which describe higher spins in (3+1) dimensions [16]. In contrast with the latter case here one does not need to impose the condition of symmetry with respect to the spinor indices. That is connected with the unidimensionality of the spinning space on the mass shell. The state vectors (41) (Bargmann–Wigner amplitudes), which obey the Dirac equation (43) for each index, are automatically symmetric in these indices. To demonstrate that we choose two arbitrary indices α_i and α_j. Then one can write

$$I = (\sigma^2 i\partial_\mu\gamma^\mu)_{\alpha\alpha_j}(i\partial_\nu\gamma^\nu)_{\alpha\alpha_i}\Psi^{(\pm)}_{\alpha_i\alpha_j} = m^2\sigma^2_{\alpha_i\alpha_j}\Psi^{(\pm)}_{\alpha_i\alpha_j}$$

in virtue of the Dirac equation (43) for both indices. On the other hand, using the properties (31) of the γ–matrices, one can write

$$I = (\sigma^2 i\partial_\mu\gamma^\mu)_{\alpha_j\alpha}(i\partial_\nu\gamma^\nu)_{\alpha\alpha_i}\Psi^{(\pm)}_{\alpha_i\alpha_j} = m^2\sigma^2_{\alpha_j\alpha_i}\Psi^{(\pm)}_{\alpha_i\alpha_j} = -I \ .$$

Thus, $\sigma^2_{\alpha_j\alpha_i}\Psi^{(\pm)}_{\alpha_i\alpha_j} = 0$, that proves the symmetry of the state vectors (41) in any two spinor indices.

4 Particles with spin one

The canonical quantization, which was done in the Sect.III for any N, reproduces the quantum mechanics of particles with spin $sN/2$ and only one polarization state. For $N = 2$ we can expect to get thus a pseudoclassical model for particles with spin one in 2+1 dimensions. Let us find a relation between such a quantum mechanics and the field theory of massive spin one particles in such dimensions. There we have two candidates, namely, Proca theory and topologically massive gauge theory of Chern-Simons field [1]. Below we are going to demonstrate that the action (1) at $N = 2$ leads to the theory of Chern-Simons particles in course of quantization. To this end let us consider first the Dirac quantization of the theory with the action (1) at $N = 2$. As was already mentioned in this scheme of quantization the

second-class constraints $\Phi_4^{(1)}$ define the Dirac brackets and therefore the commutation relations, whereas, the first-class constraints, being applied to the state vectors, define physical states without imposing explicitly any gauge conditions. For essential operators and nonzero commutation relations one can obtain in analogy with (35):

$$[\hat{x}^\mu, \hat{\pi}_\nu] = i\{x^\mu, \pi_\nu\}_{D(\Phi_4^{(1)})} = i\delta^\mu_\nu\ , \quad \mu, \nu = 0, 1, 2\ ,$$
$$[\hat{\psi}_{an}, \hat{\psi}_{bl}]_+ = i\{\psi_{an}, \psi_{bl}\}_{D(\Phi_4^{(1)})} = -\frac{1}{2}\eta_{nl}\delta_{ab}; \quad a, b = 1, 2; \quad n, l = 0, 1, 2, 3\ . \tag{45}$$

The commutation relations (45) for $\hat{x}^\mu$ and $\hat{\pi}_\nu$ can be realized in a Hilbert space $\mathcal{R}_1$ whose elements are functions dependent on x, so that $\hat{x}^\mu = x^\mu\ , \quad \hat{\pi}_\mu = -i\partial_\mu$. The commutation relations (45) for $\hat{\psi}_{an}$ one can realize in a Hilbert space $\mathcal{R}_2$, which is a Fock space constructed by means of four kinds of Fermi annihilation and creation operators $\hat{b}_n,\ \hat{b}_n^+$,

$$\hat{b}_n = \hat{\psi}_{1n} + i\hat{\psi}_{2n},\ \hat{b}_n^+ = \hat{\psi}_{1n} - i\hat{\psi}_{2n},$$
$$[\hat{b}_n, \hat{b}_l^+]_+ = -\eta_{nl},\ [\hat{b}_n, \hat{b}_l]_+ = [\hat{b}_n^+, \hat{b}_l^+]_+ = 0. \tag{46}$$

Due to the Fermi statistics of these operators $\mathcal{R}_2$ is a finite-dimensional space with the basis vectors $|0>,\ |n>,\ |nl>,\ |\widetilde{n>},\ |\widetilde{0>}$, where $|0>$ is the vacuum vector, $\hat{b}_n|0>= 0$, and

$$|n>= \hat{b}_n^+|0>,\ |nl>= \hat{b}_n^+\hat{b}_l^+|0>,\ |\widetilde{n>} = \frac{1}{6}\varepsilon^{nlcd}\hat{b}_l^+\hat{b}_c^+\hat{b}_d^+|0>,$$
$$|\widetilde{0>} = \frac{1}{24}\varepsilon^{nlcd}\hat{b}_n^+\hat{b}_l^+\hat{b}_c^+\hat{b}_d^+|0>, \quad n, l, c, d = 0, 1, 2, 3\ . \tag{47}$$

The total Hilbert space $\mathcal{R}$ of the quantum mechanics, we are constructing, is the direct product of ones $\mathcal{R}_1$ and $\mathcal{R}_2$. The states vectors $\mathbf{f}(x) \in \mathcal{R}$ can be presented in the following form

$$\mathbf{f}(x) = f(x)|0> + f^n(x)|n> + \frac{1}{2}f^{nl}(x)|nl> + \tilde{f}_n(x)|\widetilde{n>} + \tilde{f}(x)|\widetilde{0>}\ . \tag{48}$$

The physical vectors of the form (48) have to be annulled by the operators of the first-class constraints,

$$(\hat{\pi}_\mu\hat{\psi}_a^\mu + sm\hat{\psi}_a^3)\mathbf{f}(x) = 0, \tag{49}$$

$$(\varepsilon^{\mu\nu\lambda}\hat{\pi}_\mu\hat{\psi}_{a\nu}\hat{\psi}_{a\lambda} + \frac{i}{2}sm)\mathbf{f}(x) = 0, \tag{50}$$

$$(\hat{\pi}_\mu\hat{\pi}^\mu - m^2)\mathbf{f}(x) = 0. \tag{51}$$

Combining the equations (49), one can get

$$\hat{\pi}^n\hat{b}_n\mathbf{f}(x) = 0, \quad \hat{\pi}^n\hat{b}_n^+\mathbf{f}(x) = 0, \tag{52}$$

where $\hat{\pi}_n = (\hat{\pi}_\mu, m),\ n = 0, 1, 2, 3$, and combining the equations (50), we get

$$(i\varepsilon^{\mu\nu\lambda}\hat{\pi}_\mu\hat{b}_\nu^+\hat{b}_\lambda - sm)\mathbf{f}(x) = 0. \tag{53}$$

Calculating the operators $\hat{J}^\mu$ and the operator $\hat{W}$, which correspond to the dual angular momentum vector and to the Pauli-Lubanski scalar (10) in the realization (46), one can verify that the $2+1$ Poincare algebra (15) holds and

$$\hat{W} = \hat{\pi}_\mu\hat{J}^\mu = i\varepsilon^{\mu\nu\lambda}\hat{\pi}_\mu\hat{b}_\nu^+\hat{b}_\lambda\ . \tag{54}$$

Thus, the equation (53) is well known from the group theoretical analysis [12, 13] condition, which specifies the helicity s of particles. The conditions (52) (for the normalized vectors of the form (48)) lead to the following equations

$$f(x)=\tilde{f}(x)=0\ , \tag{55}$$
$$\varepsilon^{lncd}\hat{\pi}_n f_{cd}(x)=0\ , \tag{56}$$
$$\hat{\pi}_n f^{nl}(x)=0\ , \tag{57}$$

whereas the condition (53) results in

$$f^n(x)=\tilde{f}_n(x)=0\ , \tag{58}$$
$$i\hat{\pi}_n\left[-\eta_{cl}\varepsilon^{nlq3}f_{qd}(x)+\eta_{dl}\varepsilon^{nlq3}f_{qc}(x)\right]-smf_{cd}(x)=0\ . \tag{59}$$

Thus, the final form of the physical state vectors is

$$\mathbf{f}(x)=\frac{1}{2}f^{nl}(x)|nl>, \tag{60}$$

where the functions $f^{nl}(x)$ obey the equations (56, 57, 59). Let us analyze consequences of these equations in detail. First of all, it follows from the eq. (56) at $l=\mu$, that

$$f_{\mu\nu}(x)=-\frac{i}{m}(\partial_\mu f_{3\nu}-\partial_\nu f_{3\mu})\ , \tag{61}$$

then the same equation at $l=3$ is obeyed identically. The relation (61) means, in fact, that the theory can be formulated in terms of the vector field $\mathcal{F}_\mu(x)=f_{3\mu}(x)$ only. One can interpret $\mathcal{F}_\mu(x)$ as a wave function of the system in the representation of the basis $|3\mu>$ and in x-representation. The eq. (57) at $l=3$ results in the transversality condition for $\mathcal{F}_\mu(x)$,

$$\partial_\mu\mathcal{F}^\mu(x)=0\ , \tag{62}$$

whereas the same equation (57) at $l=\mu$ in combination with (62) provides the Klein-Gordon equation for $\mathcal{F}_\mu(x)$,

$$(\Box+m^2)\mathcal{F}_\mu(x)=0,\ \ \Box=\partial_\mu\partial^\mu. \tag{63}$$

Thus, the condition (51) for the whole state vector $\mathbf{f}(x)$ holds as a consequence of eq. (63). At last, we get from the equation (59) at $c=3,\ d=\mu$,

$$\partial_\lambda\varepsilon^{\lambda\mu\nu}\mathcal{F}_\nu+sm\mathcal{F}^\mu=0\ , \tag{64}$$

whereas the equations (59) at $c=\nu,\ d=\mu$ are obeyed identically as consequences of (63) and (64). The transversality condition (62) is consistent with (64). Finally, it is easy to discover that the Pauli-Lubanski operator in the representation considered has the form $(\hat{W})^\nu_\mu=\partial_\lambda\varepsilon^{\lambda\nu\alpha}\eta_{\alpha\mu}$, so that eq. (64) is the above mentioned condition, which specifies the helicity of particles.

One can see now that the equations (64) are, in fact, the field equations of the so called "self-dual" free massive field theory [14], with the Lagrangian

$$\mathcal{L}_{SD}=\frac{1}{2}\mathcal{F}^*_\mu\mathcal{F}^\mu-\frac{s}{2m}\varepsilon^{\mu\nu\lambda}\mathcal{F}^*_\mu\partial_\nu\mathcal{F}_\lambda\ . \tag{65}$$

As was remarked in [15] this theory is equivalent to the topologically massive gauge theory [1] with the Chern-Simons term. Indeed, the transversality condition (62) can be viewed as a Bianchi identity, which allows introducing gauge potentials A_μ, namely a transverse vector may be written (in topologically trivial space-time) as a curl,

$$\mathcal{F}^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = \frac{1}{2}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda} \ , \tag{66}$$

where $F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$ is the field strength. Thus, $\mathcal{F}^\mu$ appears to be the dual field strength, which is a tree-component vector in $2+1$ dimensions. Then (64) implies the following equations for $F_{\mu\nu}$

$$\partial_\lambda F^{\lambda\mu} + s\frac{m}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} = 0 \ , \tag{67}$$

which are the field equations of the topologically massive gauge theory with the Lagrangian

$$\mathcal{L}_{CS} = -\frac{1}{4}F^*_{\mu\nu}F^{\mu\nu} + s\frac{m}{4}\varepsilon^{\mu\nu\lambda}F^*_{\mu\nu}A_\lambda \ . \tag{68}$$

The theory describes particles with the mass m and spin $s = \pm 1$, having only one polarization state, what has been noted by several authors [1, 12].

One can also find a relation between the Bargmann–Wigner type realization presented in the previous section and the description of the spin one particle in terms of the vector field. Let $\Psi_{\alpha\beta}(x) = \Psi^{(+)}_{\alpha\beta}(x) + \Psi^{(-)}_{\alpha\beta}(x)$, where $\Psi^{(\pm)}(x)$ are the Bargmann–Wigner amplitudes (41) for $N = 2$. Then construct a vector field $\mathcal{F}_\mu(x)$,

$$\mathcal{F}_\mu(x) = \frac{1}{\sqrt{2}}(\sigma^2\gamma_\mu)_{\alpha\beta}\Psi_{\alpha\beta}(x) \ . \tag{69}$$

The relation between $\mathcal{F}_\mu(x)$ and $\Psi_{\alpha\beta}(x)$ is one–to–one correspondence,

$$\Psi_{\alpha\beta}(x) = \frac{1}{\sqrt{2}}(\gamma^\mu\sigma^2)_{\alpha\beta}\mathcal{F}_\mu(x) \ .$$

Contracting the Dirac equation (43) with the matrices $\sigma^2\gamma^\mu$ and the using the symmetry property of $\Psi_{\alpha\beta}(x)$, we verify that the equation (64) holds for $\mathcal{F}^\mu(x)$.

The Lagrangian (65) can be rewritten in terms of the Bargmann–Wigner amplitude $\Psi_{\alpha\beta}(x)$,

$$\mathcal{L}_{SD} = \frac{s}{2m}\overline{\Psi}_{\alpha\beta}(i\partial_\mu\gamma^\mu - sm)_{\alpha\gamma}\Psi_{\gamma\beta}, \quad \overline{\Psi}_{\alpha\beta} = \Psi^*_{\gamma\delta}\gamma^0_{\gamma\alpha}\gamma^0_{\delta\beta} \ .$$

Thus we get a new formulation of the "self–dual" theory.

It is interesting to present for comparison a pseudoclassical model, which reproduces the Proca theory after quantization. Such a model can be derived by means of direct dimensional reduction from the corresponding $3+1$ dimensional model [4, 5]. The action in $2+1$ dimensions can be written as

$$S_{Pr} = \int_0^1 \left[-\frac{z^2}{2e} - \frac{m^2}{2}e - i\sum_{a=1}^{2}\left(m\psi^3_a\chi_a + \psi_{an}\dot{\psi}^n_a\right) + \sum_{a,b=1}^{2} f_{ab}\left(\frac{i}{2}[\psi_{an},\psi^n_b] + \varepsilon_{ab}\right)\right] d\tau \ ,$$

$$z^\mu = \dot{x}^\mu - i\sum_{a=1}^{2}\psi^\mu_a\chi_a \ . \tag{70}$$

All the notations are similar to (1), the new even variables f_{ab} (f_{ab} is antisymmetric) are only introduced and ε_{ab} is two-dimensional Levi-Civita symbol. Both symmetries (2) and (3) remain for the action (70) (with the corresponding z), and instead of the gauge transformations (3) appear new ones

$$\delta x^{\mu}=0\ ,\ \ \delta e=0,\ \delta\psi_a^n=\sum_{c=1}^{2}t_{ac}\psi_c^n,\ \delta\chi_a=\sum_{c=1}^{2}t_{ac}\chi_c,\ \delta f_{ab}=\dot{t}_{ab}+\sum_{c=1}^{2}(t_{ac}f_{cb}-t_{bc}f_{ca})\ . \quad (71)$$

The classical analysis shows that the total Hamiltonian is $H^{(1)}=H+\lambda_A\Phi_A^{(1)}$ with

$$H=-\frac{e}{2}(\pi^2-m^2)+i\sum_{a=1}^{2}(\pi_\mu\psi_a^\mu+m\psi_a^3)\chi_a-\sum_{a,b=1}^{2}f_{ab}\left(\frac{i}{2}[\psi_{an},\psi_b^n]+\varepsilon_{ab}\right)\ ,$$

and all the constraints coincide with ones for the action (1) at $N=2$, only the first-class constraints $\Phi_3^{(2)}=0$ have to be replaced by

$$\Phi_3^{(2)}=\frac{i}{2}[\psi_{1n},\psi_2^n]+1=0\ . \quad (72)$$

As a result, one can perform the Dirac quantization completely in the same way as was done in the Sect.V. The only difference is connected with the different form of the first-class constraint $\Phi_3^{(2)}$. Thus, one of the conditions, which define the physical states, namely, the condition (50) has to be replaced by the condition

$$\left(i[\hat{\psi}_{1n},\hat{\psi}_2^n]+2\right)\mathbf{f}(x)=0\ \Rightarrow\left(\hat{b}_n^+\hat{b}_n+2\right)\mathbf{f}(x)=0\ . \quad (73)$$

This condition results only in the equation (58). Thus, in the case under consideration, the physical state vectors have the same form (60), where the functions $f^{nl}(x)$ obey only the equations (56-58). Taking into account the consequences of these equations, one can see that the theory can be formulated in terms of the vector field $\mathcal{F}_\mu(x)$, which obeys only the equations of transversality (62) and the Klein-Gordon equation (63), those both are equivalent to the Proca equations for the massive vector field,

$$\partial_\lambda F^{\lambda\mu}+m^2\mathcal{F}^\mu=0,\ \ F_{\mu\nu}=\partial_\mu\mathcal{F}_\nu(x)-\partial_\nu\mathcal{F}_\mu(x)\ , \quad (74)$$

which implies the Proca Lagrangian

$$\mathcal{L}_{Pr}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{m^2}{2}\mathcal{F}_\mu\mathcal{F}^\mu\ . \quad (75)$$

In the Proca theory in $2+1$ dimensions two of three components $\mathcal{F}_\mu(x)$ are independent, so that two polarization states are available. In accordance with the group theoretical analysis that means that the Proca field corresponds to a reducible spin one representation of the Poincare group in $2+1$ dimensions.

Comparing the action (1) at $N=1$ and the action (70), one can believe that the necessary reduction to only one polarization state is achieved in the pseudoclassical action (1) due to the presence of terms having a structure similar to the Chern-Simons term in the field theory action (68).

In the conclusion to this section one ought to remark that we have only demonstrated a relation between the quantum mechanics constructed in course of the quantization and the field theory in cases of spin 1/2 and 1. The same can be done by analogy in cases of spin 3/2, 2, and 5/2, for which the corresponding field theory is constructed [17]. Unfortunately, for other higher spins the problem of the field theory construction is still open. Its solution can be related with an appropriate choice of the higher spins description.

5 Weyl particles with higher spins in 3+1 dimensions

It is interesting that the model for spin 1/2 in $2+1$ dimensions ($N = 1$) is related by means of a dimensional reduction procedure to the model of the Weyl particle in $3+1$ dimensions, proposed in [6]. The action of the latter model has the form

$$S = \int_0^1 \left[-\frac{z^2}{2e} - i\psi_\mu \dot{\psi}^\mu \right] d\tau \ ,$$
$$z^\mu = \dot{x}^\mu - \varepsilon^{\mu\nu\lambda\sigma}\kappa_\nu \psi_\lambda \psi_\sigma - i\psi^\mu \chi - \frac{is}{2}\kappa^\mu \ , \tag{76}$$

where $\mu = 0, 1, 2, 3$, and $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. In the gauge $\psi^0 = 0$ one can see that among the four constraints T_μ of the model only one is independent. Thus, in fact, one can use only one component of κ^μ and all others put to be zero. In $3+1$ dimensions this violates the explicit Lorentz invariance on the classical level. However in $2+1$ dimensions it does not. So, if we make a dimensional reduction $3+1 \to 2+1$ in the Hamiltonian and constraints of the model (76), putting also $\pi_3 = m$, $\kappa_3 = \kappa$, whereas $\kappa^0 = \kappa^1 = \kappa^2 = 0$, then as a result of such a procedure we just obtain the expression for the Hamiltonian of the massive spin 1/2 particle in 2+1 dimensions and all the constraints of the latter model. In the presence of an electromagnetic field one has also to put $A_3 = 0$, $\partial_3 A_\mu = 0$ to get the same result.

Thus, one can think that an action, which describes the Weyl higher spin particles in 3+1 dimensions, can be constructed in analogy with the general action (1) in $2+1$ dimensions. Namely, to describe Weyl particles with higher (integer and half–integer) spins (helicities) one needs to transform the action (76) into the following one:

$$S = \int_0^1 \left[-\frac{z^2}{2e} - i\sum_{a=1}^{N} \psi_{a\mu} \dot{\psi}_a^\mu \right] d\tau \ ,$$
$$z^\mu = \dot{x}^\mu - \sum_{a=1}^{N} (\varepsilon^{\mu\nu\lambda\sigma}\kappa_{a\nu}\psi_{a\lambda}\psi_{a\sigma} + i\psi_a^\mu \chi_a + \frac{is}{2}\kappa_a^\mu) \ . \tag{77}$$

The hamiltonization of the theory and its quantization can be done quite similar to that for the actions (76) and (1). We describe briefly here only key formulas and steps, using the previous notations.

The primary and secondary constraints and the Hamiltonian (the latter is proportional to the secondary constraints) are

$$\Phi_1^{(1)} = P_e \ , \ \ \Phi_2^{(1)} = P_{\chi_a} \ , \ \ \Phi_3^{(1)} = P_{\kappa_a^\mu} \ , \ \ \Phi_4^{(1)} = P_{a\mu} + i\psi_{a\mu} \ ,$$
$$\Phi_1^{(2)} = \pi^2 - m^2, \ \Phi_2^{(2)} = \pi_\mu \psi_a^\mu \ , \ \ \Phi_3^{(2)} = \varepsilon^{\mu\nu\lambda\sigma}\pi_\nu \psi_{a\lambda}\psi_{a\sigma} + \frac{is}{2}\pi^\mu \ , \tag{78}$$

$$H = -\frac{e}{2}(\pi^2 - m^2) + \sum_{a=1}^{N}\left[i\pi_\mu\psi_a^\mu\chi_a - (\varepsilon_{\mu\nu\lambda\sigma}\pi^\nu\psi^{a\lambda}\psi^{a\sigma} + \frac{is}{2}\pi_\mu)\kappa_a^\mu\right] . \tag{79}$$

After the gauge fixing,

$$e + \zeta\pi_0^{-1} = \chi_a = \kappa_a^\mu = x_0 - \zeta\tau = \psi_a^0 = 0 ,$$

and transition to the variable $x_0' = x_0 - \zeta\tau$, we remain with the physical Hamiltonian

$$H = \omega = \sqrt{\vec{\pi}^2}, \quad \vec{\pi}^2 = \pi_k\pi_k ,$$

and with the variables x^k, π_k, $\psi_{a\perp}^k$, $k = 1, 2, 3$, $\pi_k\psi_{a\perp}^k = 0$, which obey the Dirac brackets

$$\begin{aligned}
&\{x^k, \pi_l\}_D = \delta_{kl} , \quad \{x^k, x^l\}_D = \frac{i}{\omega^2}\sum_{a=1}^{N}[\psi_{a\perp}^k, \psi_{a\perp}^l] , \quad \{x^k, \psi_{a\perp}^l\}_D = -\frac{1}{\omega^2}\psi_a^k\pi_l , \\
&\{\psi_{a\perp}^k, \psi_{b\perp}^l\}_D = -\frac{i}{2}\Pi_l^k(\pi)\delta_{ab} , \\
&\Pi_l^k(\pi) = \delta_{kl} - \frac{1}{\omega^2}\pi_k\pi_l .
\end{aligned} \tag{80}$$

The only first–class constraints, which are quadratic in grassmanian variables, have the form

$$\Phi_3^{(2)} = \frac{i}{2}\pi_\mu T_a , \quad T_a = -\frac{2i\zeta}{\omega}\varepsilon^{klm}\pi_k\psi_{a\perp}^l\psi_{a\perp}^m - s . \tag{81}$$

In course of quantization the variables turn out to be operators with commutation relations:

$$\begin{aligned}
&[\hat{x}^k, \hat{\pi}_l] = -i\delta_{kl} , \quad [\hat{x}^k, \hat{x}^l] = \frac{1}{\hat{\omega}^2}\sum_{a=1}^{N}[\hat{\psi}_{a\perp}^k, \hat{\psi}_{a\perp}^l] , \quad [\hat{x}^k, \hat{\psi}_{a\perp}^l] = \frac{i}{\hat{\omega}^2}\hat{\psi}_{a\perp}^k\hat{\pi}_l , \\
&[\hat{\psi}_{a\perp}^k, \hat{\psi}_{b\perp}^l]_+ = \frac{1}{2}\Pi_l^k(\hat{\pi})\delta_{ab} .
\end{aligned} \tag{82}$$

In terms of the physical time x^0 the quantum Hamiltonian is $\hat{H} = \hat{\zeta}\hat{\omega}$. A realization of the Hilbert space can be constructed similar to one was made in Sect.III. Namely,

$$\mathbf{\Psi} = \begin{pmatrix} \Psi^{(+)} \\ \Psi^{(-)} \end{pmatrix} ,$$

where each component $\Psi^{(\pm)}$ have N spinor indices, $\Psi^{(\pm)} = \Psi^{(\pm)}_{\alpha_1\ldots\alpha_N}$, $\alpha_a = 1, 2$. The operator $\hat{\zeta}$ acts as

$$\hat{\zeta}\mathbf{\Psi} = \begin{pmatrix} \Psi^{(+)} \\ -\Psi^{(-)} \end{pmatrix} .$$

Other operators have the form:

$$\begin{aligned}
&\hat{x}^k = x^k + \sum_{a=1}^{N}\left(\prod_{j=1}^{a-1}\otimes 1\right)\otimes\Delta\hat{x}^k\otimes\left(\prod_{j=a+1}^{N}\otimes 1\right) , \\
&\Delta\hat{x}^k = \frac{1}{2\hat{\omega}^2}\varepsilon^{klm}\hat{\pi}_l\Sigma^m , \quad \hat{\pi}_k = -i\partial_k ,
\end{aligned}$$

$$\hat{\psi}^k_{a\perp} = \left(\prod_{j=1}^{a-1} \otimes 1\right) \otimes \hat{\psi}^k_\perp \otimes \left(\prod_{j=a+1}^{n} \otimes \frac{1}{\hat{\omega}}\hat{\pi}_l \Sigma^l\right) \ , \quad \hat{\psi}^k_\perp = \frac{1}{2}\Pi^k_l(\hat{\pi})\Sigma^l \ ,$$
$$\hat{T}_a = \left(\prod_{j=1}^{a-1} \otimes 1\right) \otimes \hat{T} \otimes \left(\prod_{j=a+1}^{N} \otimes 1\right) \ ,$$
$$\hat{T} = \frac{1}{\hat{\omega}}\gamma^0 \Sigma^l \pi_l - s \ , \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{83}$$

Constructed the helicity operator $\hat{\mathbf{W}}$ calculated in the realization in question we get

$$\hat{\mathbf{W}} = -\frac{i}{2\hat{\pi}_0}\varepsilon_{klm}\hat{\pi}_k \hat{M}_{lm} = \sum_{a=1}^{N} \left(\prod_{j=1}^{a-1} \otimes 1\right) \otimes \hat{W} \otimes \left(\prod_{j=a+1}^{N} \otimes 1\right) \ ,$$
$$\hat{W} = -\frac{1}{2\hat{\pi}_0}\hat{\pi}_k \Sigma^k = \hat{T} + s \ .$$

Taking into account the physical states definition $\hat{T}_a \mathbf{\Psi} = 0$, we get $\hat{\mathbf{W}}\mathbf{\Psi} = (sN/2)\mathbf{\Psi}$, i.e. the quantum mechanics constructed describes massless particles with helicity $sN/2$.

The realization presented can be described in a slightly different form. Namely, the state vector $\mathbf{\Psi}$ is the Dirac multispinor:

$$\mathbf{\Psi} = \mathbf{\Psi}_{\alpha_1 \ldots \alpha_N} \ , \quad \alpha_a = 1, 2, 3, 4 \ .$$

It obeys both the Schrödinger equation

$$(i\frac{\partial}{\partial x^0} - \hat{\omega}\gamma^0)_{\alpha_a \alpha'_a} \mathbf{\Psi}_{\alpha_1 \ldots \alpha'_a \ldots \alpha_N} = 0 \ , \quad a = 1, \ldots, N \ , \tag{84}$$

and the conditions

$$(\frac{1}{\hat{\omega}}\gamma^0 \hat{\pi}_k \Sigma^k - s)_{\alpha_a \alpha'_a} \mathbf{\Psi}_{\alpha_1 \ldots \alpha'_a \ldots \alpha_N} = 0 \ , \quad a = 1, \ldots, N \ . \tag{85}$$

As a consequence of these equations $\mathbf{\Psi}$ is symmetric in all the indices. One can see that the equation (84) is the Dirac equation in Foldy–Wouthuysen representation and the equation (85) reproduce the Weyl condition. If we use the Foldy–Wouthuysen transformation

$$\mathbf{\Psi}^{(D)} = \prod_{a=1}^{N} \otimes U^\dagger \mathbf{\Psi} \ ,$$

$$U = \frac{\hat{\omega} + \gamma^k \hat{\pi}_k}{\sqrt{2}\hat{\omega}} \ , \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \ .$$

then the equations (84) and (85) for the vector $\mathbf{\Psi}^{(D)}$ appear to be

$$i\partial_\mu \gamma^\mu_{\alpha_a \alpha'_a} \mathbf{\Psi}^{(D)}_{\alpha_1 \ldots \alpha'_a \ldots \alpha_N} = 0 \ ,$$
$$(\gamma^5 - s)_{\alpha_a \alpha'_a} \mathbf{\Psi}^{(D)}_{\alpha_1 \ldots \alpha'_a \ldots \alpha_N} = 0 \ , \ \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad a = 1, \ldots, N. \tag{86}$$

Thus, we have obtained the massless Dirac equation and the Weyl condition for each index, i.e. the Bargmann-Wigner type description of higher massless spins in $3+1$ dimensions.

One can also verify that the model (1) is related to the model (76) by means of a dimensional reduction, similar to the case of spin 1/2.

Acknowledgments.

D. M. Gitman wish to acknowledge Brazilian foundations CNPq for partial financial support and I.V. Tyutin thanks the European Community Commission, which is supporting him in part under the contract INTAS-94-2317.

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